Some remarks on set theory. VI.

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Let E be a given non countable set of power $\mathfrak{m}$ and suppose that there exists a relation R between the elements of E. For any $x \in E$, let $R(x)$ denote the set of the elements $y \in E$ for which xRy holds. Two distinct elements of E, x and y, are called *independent*, if $x \notin R(y)$ and $y \notin R(x)$. A subset F of E is called *free* if F has only one element or if F has more elements and any two of them being independent. Let $\mathbf{B}$ be a system of subsets of E; then a non empty system $\mathbf{I} \subset \mathbf{B}$ is called a $\mathfrak{p}$-additive ideal, $\mathfrak{p} \leqq \mathfrak{m}$, if the sum of any system of power smaller than $\mathfrak{p}$, of elements of $\mathbf{I}$, is again a set of $\mathbf{I}$, and if $X \in \mathbf{I}$, $Y \in \mathbf{B}$, $Y \subset X$ imply $Y \in \mathbf{I}$.

We assume that $\{x\} \in \mathbf{B}$ and $\{x\} \in \mathbf{I}$ for every $x \in E$, and one of the following conditions holds for the sets $R(x)$:

(A) There is a cardinal number $\mathfrak{n} < \mathfrak{m}$ such that, for every $x \in E$, $\overline{\overline{R(x)}} < \mathfrak{n}$,

(B) E is a metric space and $d(x, R(x)) > 0$, where $d(x, R(x))$ denotes the distance of the point x from the set $R(x)$.

We deal in this paper first with the following question:

(i) *If* $\mathbf{A}$ *is a system of sets of* $\mathbf{B}-\mathbf{I}$, *does there exist a free subset* E' *of* E *such that for every* $X \in \mathbf{A}$, $X \cap E' \in \mathbf{B}-\mathbf{I}$?

This question has been studied previously in the following special cases:

a) $\mathfrak{m}$ is regular, condition (A) holds, $\mathbf{B}$ is the set of all subsets of E, $\mathbf{I}$ is the set of all subsets of E, of power less than $\mathfrak{m}$, and $\overline{\overline{\mathbf{A}}} = 1$ (then $\mathfrak{p} = \mathfrak{m}$). (See [1].)

b) $E = [0,1]$, with the ordinary metric, condition (B) holds, $\mathbf{B}$ is the set of all subsets of E, $\mathbf{I}$ is the set of all subsets of measure zero in the Lebesgue sense, and $\overline{\overline{\mathbf{A}}} = 1$.

(The answer to this question is affirmative, see [2].)

c) The same hypotheses as in b), with the only difference that **B** is the set of all subsets of [0,1] measurable in the Lebesgue sense.

(The answer to this question is generally in the negative. The answer is affirmative if $g(x) = d(x, R(x))$ is a measurable function in the Lebesgue sense, see [3], [4].)

d) $E = [0,1]$ with the ordinary metric d, **B** is a Boolean σ-algebra of subsets of [0,1] containing all subintervals of [0,1], and **I** is the set of the sets X of **B** such that $\mu(X) = 0$, where μ is a measure on **B**.[1])

(If μ is not identically zero and if there exists a function f measurable with respect to **B** and such that $0 < f(x) \leqq g(x) = d(x, R(x))$ for all $x \in [0,1]$, then there exists a free set F in **B** such that $\mu(F) > 0$ (i. e. $F \notin \mathbf{I}$). This theorem is due to P. Halmos.[2]))

In section I first we prove making use of a method of Ulam [6] the following theorem (Theorem 1): If E is a set of power $\aleph_\gamma$ with $\aleph_\gamma$ greater than $\aleph_0$ and less than the first aleph inaccessible in the weak sense, **I** is a proper $\aleph_{\lambda+1}$-additive ideal of subsets of E such that $\{x\} \in \mathbf{I}$ for every $x \in E$, and $F \notin \mathbf{I}$, then F may be decomposed into the sum of a sequence of the type $\omega_{\lambda+1}$, of mutually disjoint subsets F_ξ of E, such that $F_\xi \notin \mathbf{I}$.

We use this theorem in the proof of theorem 3.

In sections I and II a number of results is given with respect to question (i). For instance we shall prove that the answer to the problem is affirmative in the following cases:

1) If $\mathfrak{m} > \aleph_0$ is less than the first aleph inaccessible in the weak sense, **B** is the set of all subsets of E, **I** is a $\aleph_{\gamma+1}$ additive ideal ($\aleph_{\gamma+1} \leqq \mathfrak{m}$), $\bar{\bar{\mathbf{A}}} = \aleph_0$, and $\overline{\overline{R(x)}} < \aleph_0$ for every $x \in E$.

2) If E is a metric space which contains a dense subset, the power of which is less than the first aleph inaccessible in the weak sense, **B** is the set of all Borel sets of E, **I** is the σ-ideal of all sets of μ-measure zero of **B**, where μ is a measure on **B**, $\bar{\bar{\mathbf{A}}} = 1$, the condition (B) is satisfied, and also the following condition (C) holds:

(C) there is a real number $i > 0$ such that the set $\{x : g(x) \geqq i\}$ contains in **B** a subset of positive measure, where $g(x) = d(x, R(x))$.

If, for every $x \in E$, the set $R(x)$ is the complement of a sphere of E whose center is at x, then the condition (C) is not only sufficient, but also necessary for the existence of a free subset of E in **B**.

Finally, in the section III, we deal with the following question:

(ii) *Let* **K** *be a class of subsets of E. When does there exist a relation*

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1) We use the terminology of P. R. Halmos [11].

2) See his review of the paper [3] in *Math. Reviews*, 12 (1951), p. 398.

R for which the condition (A) *holds and there is no free subset $X \in \mathbf{K}$ with respect to R?*

For instance we shall prove that if $\overline{\overline{\mathbf{K}}} = \mathfrak{m}$ and every element of $\mathbf{K}$ is of power $\mathfrak{m}$, then there exists a relation R, with $\overline{\overline{R(x)}} \leqq 1$ for every $x \in E$, for which there is no free set in $\mathbf{K}$.

This result shows that the answer to the problem (i) is always negative if $\overline{\overline{\mathbf{B}-\mathbf{I}}} = \mathfrak{m}$ and every element of $\mathbf{B}-\mathbf{I}$ is of power $\mathfrak{m}$.

Notation and definitions. Throughout this paper, the symbols $\overline{\overline{F}}$ and $\bar{\beta}$ denote the cardinal number of the set F and of the ordinal number β, respectively. For any $x \in E$, let $R^{-1}(x) = \{y : x \in R(y)\}$. For any subset F of E let

$$R[F] = \bigcup_{x \in F} R(x) \qquad \text{and} \qquad R^{-1}[F] = \bigcup_{x \in F} R^{-1}(x).$$

For any cardinal number $\mathfrak{r}$ we denote by $\varphi_{\mathfrak{r}}$ the initial number of $\mathfrak{r}$, by $\mathfrak{r}^*$ the smallest cardinal number for which $\mathfrak{r}$ is the sum of $\mathfrak{r}^*$ cardinal numbers each of which is smaller than $\mathfrak{r}$, by $\mathfrak{r}^+$ the cardinal number immediately following $\mathfrak{r}$. We say that $\mathfrak{r}$ is regular if $\mathfrak{r}^* = \mathfrak{r}$ and singular if $\mathfrak{r}^* < \mathfrak{r}$. $\mathfrak{r} = \aleph_\gamma > \aleph_0$ is called inaccessible in the weak sense, if γ is a limit number and $\mathfrak{r}$ is regular.

I.

We assume in this section that the sets $R(x)$ satisfy condition (A) and $\mathbf{B}$ is the set of all subsets of E. We shall use the following

L e m m a. *Let T be a set of power $\aleph_{\alpha+1}$ (where α is a given ordinal number $\geqq 0$). There exists a system $\{A_\eta^\xi\}_{\eta<\omega_{\alpha+1}}^{\xi<\omega_\alpha}$ of subsets of T such that*

1) *$T = \bigcup_{\eta<\omega_{\alpha+1}} A_\eta^\xi$ for every $\xi < \omega_\alpha$,*

2) *$A_\eta^\xi \cap A_\zeta^\xi = 0$ for $\xi < \omega_\alpha$ and $\eta < \zeta < \omega_{\alpha+1}$,*

3) *the power of the set $T - \bigcup_{\xi<\omega_\alpha} A_\eta^\xi$ is $\leqq \aleph_\alpha$ for every $\eta < \omega_{\alpha+1}$.* (See S. Ulam [6] p. 143.)

We prove now the following

T h e o r e m 1. *Let E be a set of power $\aleph_\gamma$ with $\aleph_\gamma$ greater than $\aleph_0$ and less than the first aleph inaccessible in the weak sense, and let $\mathbf{I}$ be a proper $\aleph_{\lambda+1}$-additive ideal of subsets of E such that $\{x\} \in \mathbf{I}$ for every $x \in E$. If $B \subseteq E$ and $B \notin \mathbf{I}$, then there exists a sequence $\{B_\xi\}_{\xi<\omega_{\lambda+1}}$ of type $\omega_{\lambda+1}$, of subsets of E, such that*

(i) *$B_\xi \notin \mathbf{I}$ for every $\xi < \omega_{\lambda+1}$,*

(ii) *$B_\xi \cap B_\zeta = 0$ for $\xi < \zeta < \omega_{\lambda+1}$,*

(iii) *$B = \bigcup_{\xi<\omega_{\lambda+1}} B_\xi$.*

Proof[3]). We use transfinite induction. First we prove that our theorem is true for $\gamma=\lambda+1$. Let $\overline{\overline{E}}=\aleph_{\lambda+1}$ and $B\notin\mathbf{I}$. It is obvious that $\overline{\overline{B}}=\aleph_{\lambda+1}$. By the lemma ($\alpha=\lambda$ and $T=B$) there is a system $\{A_\eta^\xi\}_{\eta<\omega_{\lambda+1}}^{\xi<\omega_\lambda}$ of subsets of B for which 1), 2) and 3) hold. Since $B\notin\mathbf{I}$ and, by 3) $B-\bigcup_{\xi<\omega_\lambda}A_\eta^\xi\in\mathbf{I}$ for every $\eta<\omega_{\lambda+1}$, there exists for every $\eta<\omega_{\lambda+1}$ an ordinal number $\xi(\eta)<\omega_\lambda$ such that $A_\eta^{\xi(\eta)}\notin\mathbf{I}$. It follows that there is an ordinal number $\xi_0<\omega_\lambda$ and a sequence $\{\eta_\nu\}_{\nu<\omega_{\lambda+1}}$ of type $\omega_{\lambda+1}$, of the ordinal numbers $\varrho<\omega_{\lambda+1}$, such that $\xi(\eta_\nu)=\xi_0$ and $A_{\eta_\nu}^{\xi_0}\notin\mathbf{I}$ for every $\nu<\omega_{\lambda+1}$. Let $A=\{\eta:\eta<\omega_{\lambda+1}$ and $\eta\neq\eta_\nu$ if $\nu<\omega_{\lambda+1}\}$ and

$$B_\nu=\begin{cases}A_{\eta_0}^{\xi_0}\cup\left(\bigcup_{\eta\in A}A_\eta^{\xi_0}\right) & \text{for } \nu=0,\\ A_{\eta_\nu}^{\xi_0} & \text{for } 0<\nu<\omega_{\lambda+1}.\end{cases}$$

Obviously the set $\{B_\nu\}_{\nu<\omega_{\lambda+1}}$ satisfies the conditions (i), (ii) and (iii).

Let now β be a given ordinal number, $\beta>\lambda+1$, such that $\aleph_\beta$ is less than the first aleph inaccessible in the weak sense, and suppose that the theorem is true for every $\alpha<\beta$. Let $\overline{\overline{E}}=\aleph_\beta$ and $B\notin\mathbf{I}$ $(B\subseteq E)$.

If $\overline{\overline{B}}<\aleph_\beta$, then the theorem is true by the induction hypothesis. (Let $I_1\in\mathbf{I}_1$ if and only if $I_1=B\cap I$, where $I\in\mathbf{I}$. Obviously $\mathbf{I}_1$ is an $\aleph_{\lambda+1}$-additive ideal in B.)

If $\overline{\overline{B}}=\aleph_\beta$, then there are two possibilities:

a) β is an ordinal number of the first kind, i. e. $\beta=\alpha+1$,

b) β is an ordinal number of the second kind.

Case a). By the lemma ($\beta=\alpha+1$ and $T=B$) there is a system $\{A_\eta^\xi\}_{\eta<\omega_{\alpha+1}}^{\xi<\omega_\alpha}$ of subsets of B for which 1), 2) and 3) hold.

We have two subcases:

a$_1$) if $B=\bigcup_{\zeta<\omega_\alpha}C_\zeta$ is an arbitrary decomposition of B into the sum of $\aleph_\alpha$ subsets, then there is an ordinal number $\zeta_0<\omega_\alpha$ such that $C_{\zeta_0}\notin\mathbf{I}$,

a$_2$) B has a decomposition $B=\bigcup_{\zeta<\omega_\alpha}C_\zeta$ into the sum of $\aleph_\alpha$ subsets such that, for every $\zeta<\omega_\alpha$, $C_\zeta\in\mathbf{I}$.

Subcase a$_1$). For every $\eta<\omega_{\alpha+1}$ there is an ordinal number $\xi(\eta)<\omega_\alpha$ such that $A_\eta^{\xi(\eta)}\notin\mathbf{I}$. It follows that there is an ordinal number $\xi_0<\omega_\alpha$ and a sequence $\{\eta_\nu\}_{\nu<\omega_{\alpha+1}}$ of type $\omega_{\alpha+1}$, of ordinal numbers $\varrho<\omega_{\alpha+1}$, such that $\xi(\eta_\nu)=\xi_0$ and $A_{\eta_\nu}^{\xi_0}\notin\mathbf{I}$ for every $\nu<\omega_{\alpha+1}$. Let $A=\{\eta:\eta<\omega_{\alpha+1}$ and $\eta\neq\eta_\nu$

[3]) We make use of a method of Ulam [6].

if $\nu < \omega_{\lambda+1}\}$, and

$$B_\nu = \begin{cases} A^{\xi_0}_{\eta_0} \cup \left(\bigcup_{\eta \in A} A^{\xi_0}_{\eta} \right) & \text{for } \nu = 0, \\ A^{\xi_0}_{\eta_\nu} & \text{for } 0 < \nu < \omega_{\lambda+1}. \end{cases}$$

Subcase a$_2$). Let $B = \bigcup_{\zeta < \omega_\alpha} C_\zeta$ be a decomposition of B into the sum of $\aleph_\alpha$ subsets such that $C_{\zeta_1} \cap C_{\zeta_2} = 0$ for $\zeta_1 < \zeta_2 < \omega_\alpha$ and $C_\zeta \in \mathbf{I}$ for every $\zeta < \omega_\alpha$. Consider the set $D = \{C_\zeta\}_{\zeta < \omega_\alpha}$. We define an $\aleph_{\lambda+1}$-additive ideal $\mathbf{I}'$ in D as follows: Let $F \in \mathbf{I}'$ if and only if $F \subset D$ and $\bigcup_{C \in F} C \in \mathbf{I}$. Since $\overline{\overline{D}} = \aleph_\alpha < \aleph_\beta$ and $D \notin \mathbf{I}'$, there is, by the induction hypothesis, a decomposition

$$D = \bigcup_{\eta < \omega_{\lambda+1}} F_\eta$$

of D into the sum of $\aleph_{\lambda+1}$ subsets such that $F_{\eta_1} \cap F_{\eta_2} = 0$ if $\eta_1 \neq \eta_2$ and $F_\eta \notin \mathbf{I}'$ for every $\eta < \omega_{\lambda+1}$. Let

$$B_\eta = \bigcup_{C \in F_\eta} C.$$

Obviously $B_{\eta_1} \cap B_{\eta_2} = 0$ if $\eta_1 \neq \eta_2$, $B_\eta \notin \mathbf{I}$ for every $\eta < \omega_{\lambda+1}$, and

$$B = \bigcup_{\eta < \omega_{\lambda+1}} B_\eta.$$

Case b). Since $\aleph_\beta$ is less than the first aleph inaccessible in the weak sense, B has a decomposition $B = \bigcup_{\xi < \omega_\eta} C_\xi$ into the sum of $\aleph_\eta < \aleph_\beta$ subsets such that $\aleph_\lambda < \overline{\overline{C_\xi}} < \aleph_\beta$ and $C_{\xi_1} \cap C_{\xi_2} = 0$ if $\xi_1 \neq \xi_2$.

If there is an ordinal number $\xi_0 < \omega_\eta$ for which $C_{\xi_0} \notin \mathbf{I}$, then there is, by the induction hypothesis, a decomposition

$$C_{\xi_0} = \bigcup_{\zeta < \omega_{\lambda+1}} D_\zeta$$

of C_{ξ_0} such that $D_{\zeta_1} \cap D_{\zeta_2} = 0$ for $\zeta_1 \neq \zeta_2$ and $D_\zeta \notin \mathbf{I}$ for every $\zeta < \omega_{\lambda+1}$. Let

$$B_\zeta = \begin{cases} D_0 \cup \left(\bigcup_{\substack{\xi < \omega_\eta \\ \xi \neq \xi_0}} C_\xi \right) & \text{for } \zeta = 0, \\ D_\zeta & \text{for } 0 < \zeta < \omega_{\lambda+1}. \end{cases}$$

Obviously the set $\{B_\zeta\}_{\zeta < \omega_{\lambda+1}}$ satisfies the conditions (i), (ii), and (iii).

The proof of the case, when $C_\xi \in \mathbf{I}$ for every $\xi < \omega_\eta$, is similar to that of case a$_2$). Theorem 1 is proved.

Corollary 1. *If $\overline{\overline{E}} = \mathfrak{m} > \aleph_0$ is less than the first aleph inaccessible in the weak sense, then every finite measure μ,*[4] *defined for all subsets of E and vanishing for all one-point sets, vanishes identically.* (See S. Ulam [6].)

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[4]) We call a measure every extended real valued, non negative, countably additive set function $\mu(X)$ defined in a ring of subsets of E. A ring of sets is a non empty class $\mathbf{R}$ of sets such that if $E \in \mathbf{R}$ and $F \in \mathbf{R}$, then $E \cup F \in \mathbf{R}$ and $E - F \in \mathbf{R}$.

Proof. The set of all subsets F of E for which $\mu(F)=0$ is an $\aleph_1$-additive ideal $\mathbf{I}$ containing all one-point subsets of E. If μ is not identically zero, then there exists a subset F of E such that $\mu(F)\neq 0$; i. e. $\mathbf{I}$ is a proper ideal. By Theorem 1 there exists a sequence $\{F_\xi\}_{\xi<\omega_1}$ of type ω_1, of subsets of E, satisfying the conditions (i), (ii), (iii). Let H_n be the set of the ordinal numbers $\xi<\omega_1$ for which $\mu(F_\xi)>\frac{1}{n}$ $(n=1,2,\dots)$. It follows that there is a natural number n_0 such that $\overline{\overline{H}}_{n_0}=\aleph_0$. Let $\{i_n\}_{n<\omega}$ be an enumeration of H_{n_0}. By the σ-additivity of μ we have

$$\mu\Big(\bigcup_{n=1}^{\infty}F_{i_n}\Big)=\sum_{n=1}^{\infty}\mu(F_{i_n})\geqq\frac{1}{n_0}+\frac{1}{n_0}+\cdots+\frac{1}{n_0}+\cdots=\infty,$$

which is impossible since μ is finite.

Corollary 2. *If $2^{\aleph_0}$ is less than the first aleph inaccessible in the weak sense, then for every subset F of the second category of the set of real numbers E there is a sequence $\{F_\xi\}_{\xi<\omega_1}$ of type ω_1, of mutually disjoint subsets of E of the second category, such that*

$$F=\bigcup_{\xi<\omega_1}F_\xi.$$

Proof. The set $\mathbf{I}$ of all subsets of the first category of E is a σ-ideal (i. e. an $\aleph_1$-additive ideal). (See W. Sierpiński [8] p. 176.)

Corollary 3. *If $2^{\aleph_0}$ is less than the first aleph inaccessible in the weak sense and $\mu^*(F)$ is an outer measure[5]) not identically zero on the set of all subsets of the set E of real numbers such that $\mu^*(\{x\})=0$ for every $x\in E$, then for every subset F of E for which $\mu^*(F)\neq 0$, there is a sequence $\{F_\xi\}_{\xi<\omega_1}$ of the type ω_1, of mutually disjoint subsets F_ξ of E such that $\mu^*(F_\xi)\neq 0$ and*

$$F=\bigcup_{\xi<\omega_1}F_\xi.$$

Proof. The set $\mathbf{I}$ of all subsets F of E for which $\mu^*(F)=0$ is a σ-ideal. (See W. Sierpiński [8] p. 109, Proposition C_{54}.)

Theorem 2. *Let $\overline{\overline{E}}=\aleph_\gamma>\aleph_0$ and suppose that there exists a relation R between the elements of E, such that for any $x\in E$, the power of the set $R(x)=\{y: xRy\}$ is smaller than $\mathfrak{n}<\mathfrak{m}$. Let furthermore $\mathbf{I}$ be an $\mathfrak{n}^+$-additive proper ideal of E, such that $\{x\}\in\mathbf{I}$ for any $x\in E$. Then there exists a free subset E' of E, such that $E'\notin\mathbf{I}$,*

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[5]) An outer measure is an extended real valued, non negative, monotone and countably subadditive set function ν^* on the class of all subsets of E, such that $\nu^*(0)=0$.

Proof. By Theorem 1 of [5] E may be decomposed into the sum of $\mathfrak{n}$ or fewer free subsets E_ξ $(\xi < \varphi_{\mathfrak{n}})$:

$$E = \bigcup_{\xi < \varphi_{\mathfrak{n}}} E_\xi .$$

Since $\mathbf{I}$ is an $\mathfrak{n}^+$-additive proper ideal it follows the statement of Theorem 2.

Theorem 3. *Let E be a set of power $\aleph_\gamma$ with $\aleph_\gamma$ greater than $\aleph_0$ and less than the first aleph inaccessible in the weak sense, and let R be a relation between the elements of E such that for any $x \in E$ the power of the set $R(x)$ is smaller than $\aleph_0$. Let furthermore $\mathbf{I}$ be an $\aleph_{\lambda+1}$-additive proper ideal of subsets of E, such that $\{x\} \in \mathbf{I}$ for any $x \in E$. If $\{E_\xi\}_{\xi<\omega}$ is a sequence of type ω, of subsets of E, such that $E_\xi \notin \mathbf{I}$ for $\xi < \omega$, then there exists a free subset E' of E for which $E' \cap E_\xi \notin \mathbf{I}$ for every $\xi < \omega$.*

Proof. First we define by finite induction a sequence $\{F_\xi\}_{\xi<\eta}$ of subsets of E such that $F_\xi \notin \mathbf{I}$ for $\xi < \eta$, $F_{\xi_1} \cap F_{\xi_2} = 0$ if $\xi_1 \neq \xi_2$, and for every $\xi < \omega$ there is a $\nu(\xi) < \eta$ such that $F_{\nu(\xi)} \subset E_\xi$. Let $E_0 = \bigcup_{\nu<\omega_1} E_{0\nu}$ be a decomposition of E_0 satisfying Theorem 1. Since $E_{0\nu} \cap E_{0\mu} = 0$ for $\nu \neq \mu$, for every $\xi < \omega$ there at most one $\nu = \nu(\xi) < \omega_1$ such that $E_\xi - E_{0\nu(\xi)} \in \mathbf{I}$. It follows that there is an ordinal number $\nu' < \omega_1$ for which $E_\xi - E_{0\nu'} \notin \mathbf{I}$, for every $\xi < \omega$. Put $F_0 = E_{0\nu'}$. Let $\beta < \omega$ be a given ordinal number $\beta > 0$, and suppose that all sets F_ξ, where $0 \leqq \xi < \beta$, have been already defined such that $F_\xi \notin \mathbf{I}$ for $\xi < \beta$ and $F_{\xi_1} \cap F_{\xi_2} = 0$. Put $E_\xi - \bigcup_{\zeta<\xi} F_\zeta = N_\xi$ $(\xi \geqq \beta)$. Let $U = \{\xi : \beta \leqq \xi < \omega \text{ and } N_\xi \notin \mathbf{I}\}$. If $U = 0$, then we do not define F_β. In this case we put $\mu = \beta$. If $\overline{\overline{U}} = 1$, i. e. $U = \{k\}$, then let $F_\beta = N_k$ and $\eta = \beta + 1$. If $\overline{\overline{U}} > 1$, then we denote by ϱ the first element of U. Let $N_\varrho = \bigcup_{\nu<\omega_1} N_{\varrho\nu}$ be a decomposition of N_ϱ satisfying Theorem 1. Since $N_{\varrho\nu} \cap N_{\varrho\gamma} = 0$ for $\nu \neq \mu$, there is a $\nu < \omega_1$ such that $N_\xi - N_{\varrho\nu} \notin \mathbf{I}$ for every $\xi \in U$. Put $F_\beta = N_{\varrho\nu}$.

It follows from Theorem 2 that F_ξ has for every $\xi < \eta$ a free subset G_ξ such that $G_\xi \notin \mathbf{I}$. We shall now prove that there is a sequence $\{H_\xi\}_{\xi<\eta}$ of subsets of E such that $H_\xi \subset G_\xi$, $H_\xi \notin \mathbf{I}$ $(\xi < \eta)$ and $H_\xi \cap (R[H_\zeta] \cup R^{-1}[H_\zeta]) = 0$ for $\xi \neq \zeta$. The set $E' = \bigcup_{\xi<\eta} H_\xi$ obviously satisfies Theorem 2.

We define H_0 as follows. Let $G_0 = \bigcup_{\alpha<\omega_1} G_{0\alpha}$ be a decomposition of G_0 satisfying Theorem 1. There is an ordinal number $\alpha' < \omega_1$ such that $G_\xi - R^{-1}(G_{0\alpha'}) \notin \mathbf{I}$. In the opposite case there would exist for every α a natural number $\xi = \xi(\alpha)$ such that $G_{\xi(\alpha)} - R^{-1}[G_{0\alpha}] \in \mathbf{I}$. This would imply the existence of a natural number ξ' and a sequence $\{\alpha_k\}_{k<\omega}$ such that $\xi' = \xi(\alpha_k)$

for every $k<\omega$, i. e. $G_{\xi'}-R^{-1}[G_{0\alpha_k}]\in \mathbf{I}$ for every $k<\omega$. Then there would exist an element $z\in G_{\xi'}$, for which $z\in R^{-1}[G_{0\alpha_k}]$, i. e. $R(z)\cap G_{0\alpha_k}\neq 0$ for every $k<\omega$, which is a contradiction, because $\overline{\overline{R(z)}}<\aleph_0$.

Put $G'_\xi=G_\xi-R^{-1}[G_{0\alpha'}]$ $(\xi=1,2,\ldots)$. Let $G'_\xi=\bigcup_{\alpha<\omega_1} G'_{\xi\alpha}$ be a decomposition of G'_ξ satisfying Theorem 1. Further let

$$U_\alpha=\bigcup_{0<\xi<\eta} G'_{\xi\alpha}.$$

It is obvious that $U_{\alpha_1}\cap U_{\alpha_2}=0$ for $\alpha_1\neq\alpha_2$.

There is a natural number ν' for which $G_{0\alpha'}-R^{-1}[U_{\nu'}]\notin\mathbf{I}$. For if $G_{0\alpha'}-R^{-1}[U_\nu]\in\mathbf{I}$ for every $\nu<\omega$, then there would exist an element $z\in G_{0\alpha'}$ such that $z\in R^{-1}[U_\nu]$ $(\nu=0,1,2,\ldots)$ i. e. $R(z)\cap U_\nu\neq 0$ $(\nu=0,1,2,\ldots)$, which is impossible, because $\overline{\overline{R(z)}}<\aleph_0$. Put $H_0=G_{\alpha'}-R^{-1}[U_{\nu'}]$. It is obvious that

$$N_\xi=G'_{\xi\nu'}-R[H_0]-R^{-1}[H_0]\notin\mathbf{I}\qquad(\xi=1,2,\ldots).$$

We define H_1 starting from N_1 in the same way as H_0 is defined starting from the set G_0. Obviously we can continue this process for every $\nu<\eta$. Thus we obtain the sequence $\{H_\nu\}_{\nu<\eta}$ satisfying our requirement. The theorem is proved.

Corollary 4. *If $2^{\aleph_0}$ is less than the first aleph inaccessible in the weak sense, E is the set of the real numbers and R is a relation between the elements of E such that for any $x\in E$ the power of the set $R(x)$ is smaller than $\aleph_0$, then there exists a free subset E' of E, which is everywhere of the second category.*

Proof. Let $\mathbf{I}$ be the set of the subsets of E of the first category, and $\{E_\xi\}_{\xi<\omega}$ a sequence of type ω, of all intervals of E with rational endpoints, and apply Theorem 3.

Corollary 5. *Under the same hypotheses as in Corollary 4 there exists a free subset E' of E such that*

$$\mu^*(E'\cap[a,b])\neq 0$$

for every interval $[a, b]$ of E, μ^* denoting Lebesgue outer measure.

Proof. Let $\mathbf{I}$ be the set of all subsets of measure zero of E and $\{E_\xi\}_{\xi<\omega}$ a sequence of type ω, of all intervals of E with rational endpoints, and apply Theorem 3.

II.

We assume in this section that E is a metric space and condition (B) holds.

First we prove the following

Theorem 4. *Let E be the set of all real numbers and R a relation between the elements of E such that, for any $x \in E$, the power of the set $R(x)$ is smaller than $\aleph_0$. Then there exists a free subset E' of E such that E' is everywhere of the second category.*

Proof. Let (a, b) be an arbitrary interval of E and $\mathbf{A}^{(a,b)}$ the set of all subsets of (a, b) the complements of which are of the first category and F_σ. Let further $\{C_\gamma\}_{\gamma<\mathfrak{c}}$ be a wellordering of the set

$$\bigcup_{(a,b)\subseteq E} \mathbf{A}^{(a,b)}$$

of the type $\varphi_\mathfrak{c}$ (where $\mathfrak{c} = 2^{\aleph_0}$) and I_γ the interval corresponding to the set C_γ.

We consider the set $\mathbf{H}$ of all the series $H = \{a_\xi\}_{\xi<\varphi_\mathfrak{c}}$ of elements with the properties:

a) $a_\xi \in C_\xi$ or $a_\xi = 0$; $\xi < \varphi_\mathfrak{c}$;

b) if $a_\xi \neq 0$, then $a_\nu \neq 0$ for $\nu < \xi$;

c) if $a_\xi \neq 0$ and $a_\nu \neq 0$, then $a_\xi \neq a_\nu$ for $\xi < \nu$;

d) the set of the elements of the series is a free set.

For any $H \in \mathbf{H}$, let $\tilde{H}$ denote the set of the elements of H.

We say that an element $H \in \mathbf{H}$ is maximal with respect to the relation R if ν_0 is the smallest ordinal number $< \varphi_\mathfrak{c}$ such that $a_{\nu_0} = 0$ and there is no element $k \in C_{\nu_0} - R[\tilde{H}]$ such that k and the elements $\neq 0$ of H are independent or if $a_\nu \neq 0$ for every $\nu < \varphi_\mathfrak{c}$ We define the *index* of H in the first case as ν_0 and in the second case as $\varphi_\mathfrak{c}$. Let $\mathbf{H}'$ be the set of the maximal elements of $\mathbf{H}$.

We say that two series H_1 and H_2 are mutually exclusive if $\tilde{H}_1 \cap \tilde{H}_2 = 0$.

Let $\{H_\nu\}_{\nu<\eta}$ be a sequence of type $\eta < \omega_1$, of mutually exclusive elements of $\mathbf{H}'$ with indices $\delta_\nu < \varphi_\mathfrak{c}$. Then by the definition of $\mathbf{H}'$, $\overline{\overline{\tilde{H}_\nu}} < \mathfrak{c}$; consequently $\overline{\overline{R[\tilde{H}_\nu]}} < \mathfrak{c}$ for every $\nu < \eta$. Since $\eta < \omega_1$, by a well-known theorem of J. König we have

$$\overline{\overline{\bigcup_{\nu<\eta}(\tilde{H}_\nu \cup R[\tilde{H}_\nu])}} < \mathfrak{c},$$

i. e.

$$\overline{\overline{C_\gamma - \bigcup_{\nu<\eta}(\tilde{H}_\nu \cup R[\tilde{H}_\nu])}} < \mathfrak{c}$$

for every $\gamma < \varphi_c$. It follows that there is an element H_η of $\mathbf{H}'$ such that $\tilde{H}_\eta \neq 0$ and $\tilde{H}_\eta \cap \tilde{H}_\nu = 0$ for every $\nu < \eta$.

(1) $\left\{ \begin{array}{l} \text{For every } \delta < \varphi_c \text{ there is only a finite number of mutually exclusive} \\ \text{elements of } \mathbf{H}' \text{ with the same index } \delta. \end{array} \right.$

Let $\{H_n\}_{n<\omega}$ be a sequence of type ω, of mutually exclusive elements of $\mathbf{H}'$. Suppose that the series H_n $(n=1,2,\ldots)$ have the same index δ. Then the set $C_\gamma - \bigcup_{n<\omega} \tilde{H}_n - \bigcup R[\tilde{H}_n]$ is non empty and for every element z of this set $\overline{\overline{R(z)}} \geqq \aleph_0$ holds, because $R(z) \cap \tilde{H}_n \neq 0$ $(n=1,2,\ldots)$, which is a contradiction.

Supposing that every element of $\mathbf{H}'$ has an index smaller than φ_c, we can choose by (1) a sequence $\{H_\nu\}_{\nu<\omega_1}$ of mutually exclusive elements of $\mathbf{H}'$ of type ω_1 such that the indices β_ν of the series H_ν are distinct. Corresponding to every interval I_γ we choose in I_γ a subinterval I'_γ with rational endpoints. Since $\overline{\overline{\{\beta_\nu\}_{\nu<\omega_1}}} > \aleph_0$ and $\overline{\overline{\{I'_\gamma\}_{\gamma<\varphi_c}}} \leqq \aleph_0$, there is an I'_{γ_0} and a subsequence $\{\beta_{\nu_k}\}_k < \omega$ of type ω, of $Z = \{\beta_\nu\}_{\nu<\omega_1}$, such that $I'_{\beta_{\nu_k}} = I'_{\gamma_0}$ for every $k < \omega$. Obviously the complement of the set $L_{\gamma_0} = \bigcap_{k<\omega} C_{\beta_{\nu_k}}$ is of the first category with respect to I'_{γ_0}, Consequently the power of L_{γ_0} is $\mathfrak{c}$, thus

$$\overline{\overline{L_{\gamma_0} - \bigcup_{k<\omega} (\tilde{H}_{\nu_k} \cup R[\tilde{H}_{\nu_k}])}} = \mathfrak{c}$$

It follows that there is an element $z \in L_{\gamma_0} - \bigcup_{k<\omega} (\tilde{H}_{\nu_k} \cup R[\tilde{H}_{\nu_k}])$ such that $R(z) \cap \tilde{H}_{\nu_k} \neq 0$ $(k=1,2,\ldots)$ i. e. $\overline{\overline{R(z)}} \geqq \aleph_0$, which is impossible, because $\overline{\overline{R(z)}} < \aleph_0$. Thus there is a free subset E' of E such that $E' \cap C_\gamma \neq 0$ for every $\gamma < \varphi_c$. It is clear that E' is of the second category. The theorem is proved.

Theorem 5. *Let E be the set of all real numbers and R a relation between the elements of E such that for any $x \in E$ the power of the set $R(x)$ is smaller than $\aleph_0$. Then there exists a free subset E' of E such that the Lebesgue outer measure $\mu^*(E')$ of E' in every interval (a,b) is $b-a$.*

Proof. Let (a,b) be an arbitrary interval of E and $\mathbf{B}^{(a,b)}$ the set of all subsets of (a,b) of positive measure $> \frac{1}{2}(b-a)$ and G_δ. Let further $\{D_\gamma\}_{\gamma<\varphi_c}$ be a wellordering of the set

$$\dot{\bigcup_{(a,\,b)\subseteq E}} \mathbf{B}^{(a,\,b)}$$

of type φ_c, and I_γ the interval (a,b) corresponding to D_γ. We can prove completely analogously to the proof of the theorem 4 the existence of a free set E' such that

$$E' \cap D_\gamma \neq 0 \qquad (\gamma < \varphi_c),$$

if we select in every interval $I_\gamma = (a, b)$ an interval $I'_\gamma = (a', b')$ with rational endpoints such that $b' - a' > \frac{3}{4}(b - a)$. Obviously the outer measure of E' in every interval (a, b) is $b - a$.

It is easy to see by the method of the proofs of theorems 4 and 5 that the following theorem is valid too.

Theorem 6. *Let E be the set of all real numbers and R a relation between the elements of E such that for any $x \in E$ the power of the set $R(x)$ is smaller than $\aleph_0$. Then there exists a free subset E' of E such that E' is everywhere of the second category and the Lebesgue outer measure $\mu(E')$ of E in every interval (a, b) is $b - a$.*

Theorem 7. *Let E be an interval of the set of all real numbers and suppose that there exists a relation R between the elements of E. Let further $\mathbf{B}$ be a σ-algebra of subsets of E containing all subintervals of E and μ a not identically zero measure on $\mathbf{B}$. If $g(x) = d(x, R(x)) > 0$ for every $x \in E$ and if*

(C) *there exists a real number $i > 0$ such that the set $\{x\colon g(x) \geqq i\}$ contains in $\mathbf{B}$ a subset of positive μ-measure,*

then there exists in $\mathbf{B}$ a free subset of E of positive μ-measure.

If, for every $x \in E$, the set $R(x)$ is the complement of an interval of E whose center is at x, then the condition (C) *is not only sufficient, but also necessary for the existence of a free subset, of positive μ-measure, of E in $\mathbf{B}$.*

Proof. Let A be a subset of $\{x\colon g(x) \geqq i\}$ satisfying the condition (C). Let

$$x_1, x_2, \ldots, x_n, \ldots$$

be an enumeration of the set of rational numbers in E. For every element $x \in E$ and $\varepsilon > 0$ there exists an element x_{n_0} of this sequence such that $d(x, x_{n_0}) < \varepsilon$. For every $n = 1, 2, \ldots$ let $U(x_n, i)$ be the open interval of length i whose center is at x_n. It is obvious that

$$\bigcup_n U(x_n, i) = E.$$

Let $A_n = A \cap U(x_n, i)$ $(n = 1, 2, \ldots)$. Since $U(x_n, i) \in \mathbf{B}$ and $A \in \mathbf{B}$, $A_n \in \mathbf{B}$. Let $A_n^* = A_n - \bigcup_{j<n} A_j$ $(n = 1, 2, \ldots)$. Since μ is countably additive and $\mu(A) > 0$, there exists an index n' for which $\mu(A_{n'}^*) > 0$. It follows that $\mu(A_{n'}) > 0$. The set $A_{n'}$ is free, because if $x \in A_{n'}$ and $y \in R(x)$, then $d(x, y) > g(x) \geqq i$.

For every $x \in E$, let $U(x)$ be an interval whose center is at x and $R(x) = E - U(x)$. In this case condition (C) is also necessary for the existence of a free subset of positive μ-measure in $\mathbf{B}$, i. e. if there is in $\mathbf{B}$ a free subset A of E such that $\mu(A)>0$, then there exists a positive number i, for which the set $\{x:g(x)\geqq i\}$ contains in $\mathbf{B}$ a set of positive μ-measure. Suppose the contrary. Then $\mathbf{B}$ contains a free subset of positive μ-measure, but for every $i>0$ the set $\{x:g(x)\geqq i\}$ contains in $\mathbf{B}$ only such subsets F for which $\mu(F)=0$. Let α denote the diameter of the set A. Put

$$E_\alpha=\left\{x:\, g(x)\geqq\frac{\alpha}{2}\right\}.$$

By the hypothesis E_α contains in $\mathbf{B}$ only such subsets F, for which $\mu(F)=0$. Let $F_1=E_\alpha\cap A$ and $F_2=E_\alpha\cap(E-A)$. Since A is free and $R(x)=E-U(x)$ for every $x\in E$, we have $g(x)\geqq\frac{\alpha}{2}$ for every $x\in A$. Thus $F_1=A$. By the definition, $F_1\cup F_2=E_\alpha$, therefore $A=F_1\subset E_\alpha$. Since $A\in\mathbf{B}$, it follows that $\mu(A)=0$, which contradicts to $\mu(A)>0$. The theorem is proved.

Remark 1. *In general the condition* (C) *is not necessary*. Consider the interval $[0,1]$. Let μ^* and μ_* denote the Lebesgue outer and inner measure, respectively. We can define the relation R such that the interval $[0,1]$ contains a free subset of positive Lebesgue measure and

$$\mu_*(\{x:\; g(x)\geqq i\})=0$$

for any $i>0$, where $g(x)=d(x,R(x))$. We shall use the following theorem (see [7]):

The set E of the real numbers has a subset E' with the following properties:

1. for every interval (a,b) of E, $\mu^*(E'\cap(a,b))=b-a$,

2. E can be decomposed into enumerable many sets E_n $(n=1,2,\dots)$ without common points, which are all superposable by shifting the set E'.

It follows that $[0,1]$ can be decomposed into the sum of enumerable many sets S_n $(n=1,2,\dots)$ such that $\mu^*(S_n)=1$ $(n=1,2,\dots)$.

For every $x\in S_n$, let $K(x)$ be the open interval of length $\frac{2}{n}$ whose center is at x. We define R as follows. Let N be the set of rational numbers and

$$R(x)=(E-K(x))\cap N.$$

Obviously

$$g(x)=\frac{1}{n}\quad\text{for}\quad x\in S_n.$$

If $i>1$, then $V_i=\{x:g(x)\geqq i\}=0$. If $i\leqq 1$, then $V_i\subseteq V_{\frac{1}{n+1}}=S_1\cup S_2\cup\dots\cup S_{n+1}$ for some natural numbers $n>0$. We have $\mu_*(V_i)=0$ because $\mu_*(V_{\frac{1}{n+1}})=$ $=\mu^*([0,1]-V_{\frac{1}{n+1}})=0$.

It follows from the definition of R that the set U of the irrational numbers of $[0,1]$ is a free set. U is measurable and $\mu(U)=1$.

Remark 2. *It is easily seen that Theorem 7 remains true for a separable metric space.* The following counter-example shows that for non-separable metric spaces this theorem is generally not true.

Consider the following example of ALEXANDROFF [9]. Let S be the plane with the ordinary (euclidean) metric $d=d(x,y)$. We define now a new distance as follows. Let $\bar{0}$ be a given point of S, x and y two arbitrary points of S and

$$d'(x,y)=\begin{cases} d(x,y) & \text{if } \bar{0} \text{ lies on the line } xy, \\ d(x,\bar{0})+d(y,\bar{0}) & \text{if } \bar{0} \text{ does not lie on the line } xy. \end{cases}$$

Thus we obtain a new metric space S', which is not separable.

Let μ^* be the ordinary Lebesgue outer measure for the subsets of S. We define a relation R between the elements of S' as follows. If $x=\bar{0}$, then let $R(x)=0$. If $x\neq\bar{0}$, then let r be a real number for which $0<r<d(x,\bar{0})$, $E(x)=\{y\colon d'(x,y)<r\}$ and $R(x)=S-E(x)$. It follows from the definition of the distance d' that if $x,y\in S'$ $(x\neq y)$ and $\bar{0}$ does not lie on the line xy, then either $x\in R(y)$ or $y\in R(x)$ i. e. x and y are not independent. Hence each free subset of S' lies on a line containing $\bar{0}$. But for every line L, $\mu^*(L)=0$. Thus for every free subset E', $\mu^*(E')=0$.

For non-separable metric spaces we state the following

Theorem 8. *Let E be a metric space. Suppose that E contains a dense subset, the power of which is less than the first aleph inaccessible in the weak sense. Let μ be a σ-finite measure on the set* $\mathbf{B}$ *of all Borel subsets which is not identically zero. If $g(x)=d(x,R(x))>0$ for every $x\in E$ and if condition* (C) *holds, then there exists in* $\mathbf{B}$ *a free subset of positive μ-measure of E.*

If, for every $x\in E$, the set $R(x)$ is the complement of an sphere of E whose center is at x, then the condition (C) *is not only sufficient, but also necessary for the existence of a free subset of positive μ-measure of E in* $\mathbf{B}$.

Proof. If μ is a σ-finite measure on the set of all Borel subsets of E and E contains a dense subset, the power of which is less than the first aleph inaccessible in the weak sense, then there exists a decomposition

$$E=N\cup M$$

of E into two mutually disjoint sets such that $\mu(N)=0$ and M is separable (where N is the sum of all open subsets of μ-measure zero of E) (see [10]). It is clear that μ is not identically zero on M, since $\mu(N)=0$ and

$$\mu(N)+\mu(M)=\mu(E)\neq 0.$$

Let X be an arbitrary Borel subset of E. Since $X \cap M = X - N$ is a Borel subset of E,

$$\mu(X \cap M) = \mu(X) - \mu(N) = \mu(X).$$

Let $\mathbf{B}'$ be the set of all sets of the form $X \cap M$, where $X \in \mathbf{B}$, and let $\nu(X) = \mu(X)$ for $X \in \mathbf{B}'$. Hence, if the set $\{x : g(x) \geqq i\}$ contains in $\mathbf{B}$ a set of positive μ-measure, then it contains in $\mathbf{B}'$ a set of positive μ-measure too. Since $\mathbf{B}' \subseteq \mathbf{B}$, the converse of this statement is also true. Thus, it is sufficient to prove the theorem for M, $\mathbf{B}'$ and ν, instead of E, $\mathbf{B}$ and μ. Since M is a separable metric space and $\mathbf{B}'$ is a σ-algebra and ν is not identically zero measure on $\mathbf{B}'$, the theorem is true for M, $\mathbf{B}'$ and ν. Thus the theorem is true for E, $\mathbf{B}$ and μ too.

III.

We deal in this section with the problem (ii).

Theorem 9. *Let E be a set of power $\mathfrak{m} \geqq \aleph_0$ and $\mathbf{K}$ a class of power $\mathfrak{m}$, of subsets of E of power $\mathfrak{m}$. There exists a relation R between the elements of E such that for every $x \in E$ the power of the set $R(x)$ is $\leqq 1$ and there is no free subset X in $\mathbf{K}$ with respect to R.*

Proof. Let

$$B_0, B_1, \ldots, B_\omega, \ldots, B_\xi, \ldots \qquad (\xi < \varphi_{\mathfrak{m}})$$

be a wellordering of $\mathbf{K}$ of the type $\varphi_{\mathfrak{m}}$. Since $\overline{\overline{B_\xi}} = \mathfrak{m}$ for every $\xi < \varphi_{\mathfrak{m}}$, there exist two sequences $\{x_\xi\}_{\xi < \varphi_{\mathfrak{m}}}$ and $\{y_\xi\}_{\xi < \varphi_{\mathfrak{m}}}$ such that

1. $x_\xi \in B_\xi$ and $y_\xi \in B_\xi$ for every $\xi < \varphi_{\mathfrak{m}}$,
2. $x_\xi \neq x_\zeta$ and $y_\xi \neq y_\zeta$ for $\xi < \zeta < \varphi_{\mathfrak{m}}$,
3. $x_\xi \neq y_\xi$ for every $\xi < \varphi_{\mathfrak{m}}$.

We define R as follows: let $R(x_\xi) = \{y_\xi\}$ for every $\xi < \varphi_{\mathfrak{m}}$, and if $x \neq x_\xi$ $(\xi < \varphi_{\mathfrak{m}})$, then let $R(x) = \{x_0\}$. It is obvious that the sets B_ξ are not free.

Corollary 6. *Let E be the set of all real numbers. There exists a relation R between the elements of E such that for every $x \in E$ the power of the set $R(x)$ is $\leqq 1$ and there is no perfect free subset of E.*

Corollary 7. *Let E be the set of all real numbers. There exists a relation R between the elements of E such that for every $x \in E$ the power of the set $R(x)$ is $\leqq 1$ and there is no free Borel subset of E of power $2^{\aleph_0}$.*

Theorem 10. *Let E be a set of power $\mathfrak{m} \geqq \aleph_0$ and $\mathbf{K}$ a set of power $\mathfrak{m}$, of mutually disjoint non empty subsets of E. There exists a relation R between the elements of E, such that, for every $x \in E$ the power of the set $R(x)$ is $\leqq 1$ and there is no such free set which has non empty intersection with every element of $\mathbf{K}$.*

Proof. Let

$$B_0, B_1, \ldots, B_\omega, \ldots, B_\xi, \ldots \qquad (\xi < \varphi_{\mathfrak{m}})$$

be a wellordering of $\mathbf{K}$ of the type $\varphi_{\mathfrak{m}}$. Let further

$$x_0, x_1, \ldots, x_\omega, \ldots, x_\xi, \ldots \qquad (\xi < \varphi_{\mathfrak{m}})$$

be a wellordering of E of the type $\varphi_{\mathfrak{m}}$. Obviously, we may assume that $x_\xi \notin B_\xi$. We define R as follows: let

$$R^{-1}(x_\xi) = B_\xi.$$

Let F be a set which has non empty intersection with every element of $\mathbf{K}$:

$$F \cap B_\xi \neq 0 \qquad (\xi < \varphi_{\mathfrak{m}}).$$

Let $x \in F$. There is an ordinal number $\eta < \varphi_{\mathfrak{m}}$ such that $x = x_\eta$. Since $R^{-1}(x) = B_\eta$, we have $b_\eta R x$ for every $b_\eta \in B_\eta \cap F$. It follows that x and b_η $(x \neq b_\eta)$ are not independent, because $x \in R(b_\eta)$. The theorem is proved.

Corollary 8.[6]) *If E is the set of all real numbers, then there exists a relation R between the elements of E such that, for every $x \in E$, the power of the set $R(x)$ is $\leqq 1$ and there is no free subset, the complement of which is totally imperfect.*

Proof. Let $\mathbf{K}$ be a set of power $2^{\aleph_0}$ of non empty mutually disjoint perfect subsets of E, T a set the complement CT of which is totally imperfect, and $K \in \mathbf{K}$. Since the set CT does not contain K, $K \cap T \neq 0$. The corollary is proved.

Finally we prove

Theorem 11. *Let E be a set of power $\mathfrak{m} \geqq \aleph_0$ and $\mathbf{K}$ a class of power $\mathfrak{g} < \mathfrak{m}$, of mutually exclusive subsets of power $\mathfrak{m}$ of E. If R is a relation between the elements $x \in E$ for which the condition* (A) *holds, i. e.* $\overline{\overline{R(x)}} < \mathfrak{n} < \mathfrak{m}$ *for every $x \in E$, then there exists a free subset E' of E such that, for every $K \in \mathbf{K}$,*

$$\overline{\overline{K \cap E'}} = \mathfrak{m}.$$

Proof. Let

$$K_0, K_1, \ldots, K_\omega, K_{\omega+1}, \ldots, K_\xi, \ldots \qquad (\xi < \varphi_{\mathfrak{g}})$$

be a wellordering of $\mathbf{K}$ of the type $\varphi_{\mathfrak{g}}$. We assume first that $\mathfrak{m}$ is regular. We consider the set $\mathbf{M}$ of the matrices

$$M = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1\xi} & \ldots \\ a_{21} & a_{22} & \ldots & a_{2\xi} & \ldots \\ \vdots & \vdots & & \vdots & \\ a_{\eta 1} & a_{\eta 2} & \ldots & a_{\eta\xi} & \ldots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

[6]) S. Marcus has found independently the results of our corollaries 6 and 8.

of elements with the properties:

1. $a_{\eta\xi} \in K_\xi$ or $a_{\eta\xi} = 0$, $\eta < \varphi_{\mathfrak{m}}$ and $\xi < \varphi_{\mathfrak{g}}$,
2. if $a_{\eta\xi} \neq 0$, then $a_{\nu\mu} \neq 0$ for $\nu = \eta$ and $\mu < \xi$ or $\nu < \eta$ and $\mu < \varphi_{\mathfrak{g}}$,
3. if $a_{\nu\mu} \neq 0$ and $a_{\eta\mu} \neq 0$, then $a_{\nu\mu} \neq a_{\eta\mu}$ for $\nu \neq \eta$,
4. the set of the elements of the matrix is a free set.

For any $M \in \mathbf{M}$, let $\tilde{M}$ denote the set of the elements of M.

We say that an element $M \in \mathbf{M}$ is *maximal with respect to the relation* R if μ_0 and ν_0 are the smallest ordinal numbers $< \varphi_{\mathfrak{g}}$ such that $a_{\mu_0\nu_0} = 0$ and there is no element $k \in K_{\nu_0} - R[\tilde{M}]$ such that k and the elements $\neq 0$ of the matrix M are independent or if $a_{\mu\nu} \neq 0$ for every $\mu < \varphi_{\mathfrak{m}}$ and $\nu < \varphi_{\mathfrak{g}}$. We define the *index* of M in the first case as ν_0 and in the second case as $\varphi_{\mathfrak{g}}$. Let $\mathbf{M}'$ be the set of the maximal elements of $\mathbf{M}$.

We say that two matrices M_1 and M_2 are mutually exclusive if $\tilde{M}_1 \cap \tilde{M}_2 = 0$.

Let $\{M_\nu\}_{\nu<\eta}$ be a sequence of type $\eta < \varphi_{\mathfrak{m}}$, of mutually exclusive elements M_ν of $\mathbf{M}'$ with indices $\delta_\nu < \varphi_{\mathfrak{g}}$, Then by the definition of $\mathbf{M}'$, $\overline{\overline{\tilde{M}_\nu}} < \mathfrak{m}$, consequently $\overline{\overline{R[\tilde{M}_\nu]}} < \mathfrak{m}$ for every $\nu < \eta$, because $\overline{\overline{R(x)}} < \mathfrak{n} < \mathfrak{m}$.

Since $\mathfrak{m}$ is regular,

$$\overline{\overline{\bigcup_{\nu<\eta} (\tilde{M}_\nu \cup R[\tilde{M}_\nu])}} < \mathfrak{m}$$

i. e.

$$\overline{\overline{K_\gamma - \bigcup_{\nu<\eta} (\tilde{M}_\nu \cup R[\tilde{M}_\nu])}} < \mathfrak{m},$$

for every $\gamma < \varphi_{\mathfrak{g}}$. It follows that there is an element $M_\eta \in \mathbf{M}'$ such that $\tilde{M}_\eta \neq 0$ and $\tilde{M}_\eta \cap \tilde{M}_\nu = 0$ for every $\nu < \eta$.

(2) $\left\{ \begin{array}{l} \text{For every } \delta < \varphi_{\mathfrak{g}} \text{ there are less than } \mathfrak{n} \text{ mutually exclusive elements} \\ \text{of } \mathbf{M}' \text{ with the same index } \delta. \end{array} \right.$

Let $\{M_\nu\}_{\nu<\varphi_{\mathfrak{n}}}$ be a sequence of the type $\varphi_{\mathfrak{n}}$, of mutually exclusive elements M_ν of $\mathbf{M}'$ with the same index δ. Then the set

$$K_\delta - \bigcup_{\nu<\varphi_{\mathfrak{n}}} (\tilde{M}_\nu \cup R[\tilde{M}_\nu])$$

is non empty and, for every element z of this set, $\overline{\overline{R(z)}} \geqq \mathfrak{n}$ because, by the definition of $\mathbf{M}'$, $R(z) \cap \tilde{M}_\nu \neq 0$ for $\nu < \varphi_{\mathfrak{n}}$, which is a contradiction. Thus (2) is proved.

Supposing that every element M of $\mathbf{M}'$ has an index smaller than $\varphi_{\mathfrak{g}}$, we can now define by transfinite induction a sequence $\{M_\nu\}_{\nu<\varphi_{\mathfrak{m}}}$ of mutually exclusive elements of $\mathbf{M}'$ of the type $\varphi_{\mathfrak{m}}$. Since $\mathfrak{g} < \mathfrak{m}$ and $\mathfrak{m}$ is regular, there exists a subset, of power $\mathfrak{m}$, of $\mathbf{M}'$ with the same index $< \varphi_{\mathfrak{g}}$, which contradicts to (2). Thus there exists a matrix of index $\varphi_{\mathfrak{g}}$. It is obvious that the set of elements of this matrix satisfies the requirement of the theorem. Thus the theorem is true, if $\mathfrak{m}$ is regular.

Consider now the case when $\mathfrak{m}$ is singular[6]). We assume that the generalised continuum hypothesis is true. Let

$$\mathfrak{m}=\sum_{\xi<\varphi_{\mathfrak{m}^*}} \mathfrak{m}_\xi$$

be a decomposition of $\mathfrak{m}$ such that

1) $\mathfrak{m}_\xi$ is regular for every $\xi<\varphi_{\mathfrak{m}^*}$, 2) $\mathfrak{m}_\xi<\mathfrak{m}_\zeta$ for $\xi<\zeta<\varphi_{\mathfrak{m}^*}$,

3) $\mathfrak{m}_\xi>\max\{\mathfrak{g},\mathfrak{n},\mathfrak{m}^*\}$, 4) $2^{\sum_{\zeta<\xi}\mathfrak{m}_\zeta}<\mathfrak{m}_\xi$ for every $\xi<\varphi_{\mathfrak{m}^*}$

Let further

$$K_\nu=\bigcup_{\xi<\varphi_{\mathfrak{m}^*}} K_{\nu\xi} \qquad (\nu<\varphi_{\mathfrak{g}})$$

be a decomposition of K_ν into mutually exclusive subsets of K_ν such that $\overline{\overline{K_{\nu\xi}}}=\mathfrak{m}_\xi$.

By the first part of the theorem, there exists a free subset L_ξ of E for every $\xi<\varphi_{\mathfrak{m}^*}$ such that

$$\overline{\overline{L_\xi\cap K_{\nu\xi}}}=\mathfrak{m}_\xi$$

for every $\nu<\varphi_{\mathfrak{g}}$. Omit for $\xi<\eta$ all the elements of $R[L_\xi]$ from L_η. Thus we get the sets

$$L'_\eta=L_\eta-\bigcup_{\xi<\eta} R[L_\xi].$$

By 1) and 3), $\overline{\overline{\bigcup_{\xi<\eta} R[L_\xi]}}<\mathfrak{m}_\eta$, thus the power of the set L'_η is $\mathfrak{m}_\eta$ and $\overline{\overline{L'_\eta\cap K_{\nu\eta}}}=\mathfrak{m}_\eta$ for every $\nu<\varphi_{\mathfrak{g}}$. Obviously

$$R[L'_\xi]\cap(\bigcup_{\eta\geqq\xi} L'_\eta)=0.$$

Let

$$L'_{\nu\xi}=L'_\xi\cup K_{\nu\xi} \qquad (\nu<\varphi_{\mathfrak{g}},\ \xi<\varphi_{\mathfrak{m}^*}).$$

We want to construct sets $L''_{\nu\xi}$ of power $\mathfrak{m}_\xi$ which satisfy

$$R[L''_{\nu\xi}]\cap(\bigcup_{\varkappa<\nu}\bigcup_{\eta<\xi} L''_{\varkappa\eta})=0. \tag{3}$$

But then clearly

$$R[\bigcup_{\nu<\varphi_{\mathfrak{g}}}\bigcup_{\xi<\varphi_{\mathfrak{m}^*}} L''_{\nu\xi}]\cap[\bigcup_{\nu<\varphi_{\mathfrak{g}}}\bigcup_{\xi<\varphi_{\mathfrak{m}^*}} L''_{\nu\xi}]=0,$$

i. e. the set $\bigcup_{\nu<\varphi_{\mathfrak{g}}}\bigcup_{\xi<\varphi_{\mathfrak{m}^*}} L''_{\nu\xi}$ is free and satisfies the requirement of the theorem. Thus we only have to construct $L''_{\nu\xi}$. Consider the sets $L'_{\nu\xi}$ and

[6]) The proof is due to A. HAJNAL.

$L_\xi^* = \bigcup_{\nu<\varphi_{\mathfrak{n}}} \bigcup_{\zeta<\xi} L'_{\nu\zeta}$ ($\xi < \varphi_{\mathfrak{m}^*}$). Let $N[L_\xi^*]$ denote the set of all subsets of L_ξ^* of the power $< \mathfrak{n}$. By 3) $\overline{\overline{N[L_\xi^*]}} < \mathfrak{m}_\xi$. It follows that there exists a subset $H_{\nu\xi}$ of power $\mathfrak{m}_\xi$ of $L'_{\nu\xi}$ and an element $N_{\nu\xi}$ of $N[L_\xi^*]$ such that $L_\xi^* \cap R[H_{\nu\xi}] = N_{\nu\xi}$. Let

$$U = \bigcup_{\nu<\varphi_{\mathfrak{n}}} \bigcup_{\xi<\varphi_{\mathfrak{m}^*}} N_{\nu\xi}.$$

Obviously $\overline{\overline{U}} \leqq \mathfrak{n}\mathfrak{g}\mathfrak{m}^* < \mathfrak{m}_0$. Let $L''_{\nu\xi} = H_{\nu\xi} - U$ ($\nu < \varphi_{\mathfrak{n}}$ and $\xi < \varphi_{\mathfrak{m}^*}$). These sets obviously satisfy the condition (3). The theorem is proved.

References.

[1] S. Piccard, Sur un problème de M. Ruziewicz, de la théorie des relations, *Fundamenta Math.*, **29** (1937), 5—9.

[2] G. Fodor, On a theorem in the theory of binary relations, *Compositio Math.*, **8** (1951), 250.

[3] G. Fodor, On two problems concerning the theory of binary relations, *Publicationes Math. Debrecen*, **1** (1949—50), 199—200.

[4] G. Fodor, On a problem concerning the theory of binary relations, *Nieuw Archief voor Wiskunde*, **23** (1951), 247—248.

[5] G. Fodor, Proof of a conjecture of P. Erdős, *Acta Sci. Math.*, **14** (1952), 219—227.

[6] S. Ulam, Zur Maßtheorie in der allgemeinen Mengenlehre, *Fundamenta Math.*, **16** (1930), 140—150.

[7] W. Sierpiński, Sur un ensemble non mesurable, *Tôhoku Math. Journal*, **12** (1914), 205 - 208.

[8] W. Sierpiński, *Hypothèse du continu* (Warszawa—Lwów, 1934).

[9] П. С. Александров, Введение в общую теорию множеств и функции (Москва, 1948).

[10] E. Marczewski—R. Sikorski, Measures in non-separable metric spaces, *Colloquium Math.*, **1** (1947—1948), 133—139.

[11] P. R. Halmos, *Measure theory* (New York, 1950).

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